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Transversal mass transfer and shear stress formation during rapid gravity flow of a granular medium

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Abstract: The micro structural models for shear stress generation during rapid gravity flow of granular materials on a rough chute are discussed. The mechanism of the shear stress formation, taking into account the tangential impulse formed under transversal mass transfer of particles, is suggested. The analogy between granular media during rapid shear deformation and dense gases is used to develop the suggested mechanism on the basis of kinetic theory. The total shear stress is determined as the sum of the stress components induced by collisions, transversal mass transfer and contact interactions of uniform cohesionless inelastic spherical particles. The mathematical models describing the components of shear kinetic stresses are developed as the functions of particle properties, structural and kinematical gravity flow characteristics. The equations of impulse and energy conservation in the course of rapid gravity flow of uniform cohesionless particles are formulated. A variant of the formulation of boundary conditions at the flow bottom is proposed for mathematical modeling of the dynamics of rapid gravity flows of granular materials on a rough chute. The variant assumes the displacement of the area with the most intense shear rate inside the flow and into its layers adjacent to the rough chute surface.

Keywords: granular medium; rapid gravity flow; shear stress; transversal mass transfer; shear rate; solid volume fraction.

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Поперечный массоперенос и механизм формирования сдвиговых напряжений при быстром гравитационном течении зернистой среды

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Аннотация: Обсуждаются микроструктурные модели генерирования сдвиговых напряжений при быстром гравитационном течении зернистых материалов на шероховатом скате. Излагается механизм генерирования сдвигового напряжения, учитывающий тангенциальные импульсы, формируемые под действием поперечного массопереноса частиц. Предложенный механизм базируется на аналогии зернистой среды при быстром сдвиге и плотным газом и описывается с использованием основных положений молекулярно-кинетической теории. Общее напряжение сдвига определяется как сумма его составляющих, обусловленных столкновениями, поперечным массопереносом и контактными взаимодействиями однородных несвязных негладких сферических частиц. Разрабатываются математические модели, описывающие компоненты сдвиговых кинетических напряжений как функции свойств частиц, структурных и кинематических характеристик гравитационного потока. Формулируются уравнения сохранения импульсов и энергии при быстром гравитационном течении однородных несвязных частиц. Предлагается вариант формулировки граничных условий у основания потока при математическом моделировании динамики быстрых гравитационных течений зернистых материалов на шероховатом скате. Вариант предполагает смещение области с наиболее интенсивной скоростью сдвига внутрь потока, в его слои, прилегающие к поверхности ската.

Ключевые слова: зернистая среда; быстрое гравитационное течение; сдвиговое напряжение; поперечный массоперенос; скорость сдвига; объемная доля твердой фазы.

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1. Introduction

One of the main objects studied in physics of soft condensed matter is granular media [1, 2]. Such attention to granular materials is explained not only by the global nature of their distribution in the environment and technological processes, but also their pronounced mesoscopic properties. The presence of such properties leads to the fact that, depending on the scale of the object, the volume of its constituent set of particles and the dynamic conditions of their interaction, granular materials can be in fundamentally different states and demonstrate properties characteristic of a solid, non-Newtonian liquid or gas.

The set of properties characteristic of the listed phase states of substances can be observed, for example, in the heap of a granular material on a rough base under conditions limited only by gravity action. Depending on the angle of inclination of the rough base and the initial conditions, the behavior of elements in the granular medium will exhibit a formal analogy with a solid (at angles of inclination less than the angle of internal friction), gas (at angles of inclination greater than the angle of repose) or liquid (at some intermediate values of the angle of inclination).

The states of the granular medium, formally similar to liquid or gas, appear during shear movement of particles. Fundamental differences in the properties of the shear flow of the granular medium in liquid and gas states are in different mechanisms of shear stress generation and the degree of the dilatancy effect (increase in the volume fraction of voids). At angles of inclination of a rough base that exceed the angle of repose of the material, shear flows are characterized by significant effects of dilatancy, and shear stresses are generated mainly due to the transmission of impact pulses through the shear surface. In this case, the dependence of the dilatancy on the shear rate is clearly manifested, and when shear deformations disappear, the dilatancy decreases abruptly. This kind of granular media flows, accompanied by a relatively high rate of deformation, is usually called [3, 4] rapid shear flows or, as in the case under consideration, rapid gravity flows.

In shear flows at a low deformation rate stresses are generated mainly under the influence of friction forces arising on the shear surface during sliding and rolling of particles relative to each other. In this case, some moderate increase in dilatancy with an increase in the shear rate is observed, but in contrast to the rapid shear flow, with the disappearance of the shear deformation, there is no abrupt decrease in dilatancy. Due to the complex of the listed properties, shear flows of granular media with a low deformation rate are often called quasi-plastic.

Another characteristic difference between rapid and quasi-plastic shear flows is the duration of particle contacts during their interactions. Under conditions of rapid shear, the contacts of the particles are predominantly point-like and flowing rapidly, while, during quasi-plastic shear, the contacts of the particles are relatively long and in some way distributed over their surface in the form of contact lines.

The rapid shear flow of granular media along a rough chute, the angle of inclination of which is close to or slightly exceeds the angle of repose of the material, is the most common case of the flow of granular materials initiated by gravity action. Gravity displacements of granular materials, accompanied by rapid shear deformation, largely determine the kinetics of a large number of natural phenomena occurring on the Earth surface (avalanches, mudflows, rockfalls, sand expansion, underwater currents of solid granular rocks). In technological processes associated with the processing and use of a dispersed solid phase (chemical and biocatalysis, thermal and moisture treatment, drying, grinding, granulation, mixing, separation, etc.), rapid shear deformations create conditions for intensifying transfer phenomena in working media. Intensive transfer of energy, impulse and matter directly affects the parameters of technological flows. Adequate consideration of the transfer phenomena caused by the interaction of particles during the rapid gravity flow of granular materials is relevant for technological processes and auxiliary operations of dosing, loading and storage of various industries and the agro-industrial complex.

To predict transfer phenomena, it is necessary to have detailed information on the structural and kinematic parameters of gravity flows. Practice and research results show that values of the solid phase concentration and shear rate averaged over the flow volume do not provide conditions for adequate prediction of transfer phenomena in the flow, including the distribution of nonuniform particles [5–7].

Despite the apparent simplicity, the rapid shear flow of granular media on a rough chute is characterized by a set of parameters which create serious obstacles to the mathematical modeling of the flow dynamics (velocity profiles and distribution of solid phase concentration) [8]. For example, the problem of determining the boundary conditions and specifying the mechanism for generating shear stresses, especially in conditions of thin-layer gravity flows, is still relevant. According to the authors of this article [8], the formulation of the boundary condition at the surface of a rough chute should be closely related to the mathematical description of the mechanism for generating shear stresses within the flow, and take into account the specifics of the mechanism under conditions of edge effects.

Significant progress in the mathematical description of the dynamics of such flows could be provided by reliable experimental information on the microstructural and kinematic parameters of the flow. However, obtaining detailed experimental information on the local values of the structural and kinematic parameters of rapid gravity flows is limited by the extremely high response of the flow to internal probing [8, 9].

Rapid gravity flows of granular materials are the object of experimental and theoretical research in a large number of works, for example [3–5, 10–13]. At the same time, in many works, for example [6,8], it is indicated that, despite the great attention to the research object, a set of constitutive relations is insufficient for the development of a general model of the dynamics of rapid gravity flows. The constitutive relations providing the ability to predict shear and normal stresses in the gravity flow are of paramount importance. However, there is still a high degree of uncertainty in the formulation of this kind of constitutive relations. The problem of their formulation is explained by the complexity of determining the characteristics of the dynamic contact of particles, the kinetic energy of their chaotic movements, transverse mass transfer and rotation as a function of the angle and collision velocity of particles under certain hydrodynamic conditions of their contact (shear rate and solid volume fraction).

A large number of works [3, 10–12] are devoted to the formulation and analysis of constitutive relations for shear stresses during rapid shear deformations of granular materials. The methodological approaches used in the works for the formulation of constitutive relations can be conditionally divided into two groups: based on continual and microstructural theories [14]. In the general case, in accordance with the continual theories, shear stresses are expressed as the sum of two components: dependent and independent of the shear rate [10].

The stress component, independent of the shear rate, determines the conditions for the existence of the flow. This component establishes the relationship between shear and normal stress, thereby revealing a formal agreement with the Coulomb-Mohr theory. The stress component, which depends on the shear rate, reflects the dependence of the magnitude of the shear and normal stresses on the deformation rate.

The development of the continual approach for describing the dynamics of the rapid gravity flow is carried out by refining the constitutive relations for calculating the stress components. The main problems of using continual theories are the complexity of expressing the dependence of both stress components on the volume fraction of the solid particles, determining the physically justified range of chute angle values in which the regime of steady-state rapid shear flow is achieved, and identifying the dependence of shear and normal stresses on the shear deformation rate. To solve the complex of the listed problems of the continual theories, experimental data and analysis results of the shear flow regularities using microstructural theories, for example, methods of statistical mechanics [3, 10, 15, 16], are widely used.

In accordance with the continual theories, such stresses are taken into account by their component, which depends on the deformation rate. Special attention to the mentioned mechanism of stress formation is explained by its dominant value in the regime of the developed rapid gravity flow of cohesionless granular materials. The stress component independent of the shear rate dominates at relatively low deformation rates and a high volume fraction of the solid particles (quasi-plastic flows), while, at alternative flow parameters, the dominant role in the stress formation passes to the component that depends on the deformation rate.

As for microstructural theories, the pioneering work that gave the initial impetus to research in this direction was the work of Bagnold published in 1954 [17]. Experimental and theoretical studies carried out in the framework of this work allowed the author to detect the presence of the so-called “dispersion” pressure in the rapid shear flow of solid particles suspended in a liquid. The corresponding pressure is due to the presence of inertia in the colliding particles in the shear flow and the generation of the flow of impact pulses through the shear surface. Analyzing the impulse transfer process between particles of two adjacent layers interacting with each other through the shear surface, the author came to the conclusion that both normal and shear stresses in the rapid shear flow were proportional to the square of the shear rate. The quadratic dependence of the stress on the shear rate is explained by the fact that its values are proportional to the value of the pulse flow, which is determined by their frequency and the value of a single pulse. In particular, for the case of a two-dimensional shear flow in the Cartesian coordinate system x (in the shear direction) and y (the direction of the normal to the shear surface), the shear stress τ_{yx} is determined by the following expression

$$\tau_{yx}^{\text{col}} = p_y f \Delta M_x^{\text{col}}, \tag{1}$$

where p_y is the number of particles per unit area of the shear surface, the normal to which has the y-direction; f is the frequency of collisions of a particle with particles of an adjacent layer; ΔM_x^{col} is the component of a single impact impulse in the shear direction x.

Due to the linear dependence of the modulus and frequency of pulses in expression (1) on the shear rate, the flux of pulses, and accordingly the stress, in the shear flow of particles are presented [17] as functions depending on the second degree of the shear rate. Thus, according to Bagnold [17], the stresses generated in the rapid shear flow of particles are determined by their size, mass, concentration in the flow, restitution coefficient in a binary collision, and shear rate. This follows from the author’s assumption about the dominant role of the translational component of the particle displacement velocity in the shear direction. However, the real mechanism of particle interaction in the rapid shear flow turns out to be more complicated. Under the conditions of a shear flow, the colliding particles, in addition to their averaged translational velocity, acquire fluctuation velocity randomly distributed in space in translational displacements and rotations, and are also involved in transverse mass transfer.

Obviously, these features in the nature of mutual displacements of particles significantly affect the mechanism of stress generation in the rapid shear flow. Fluctuations of particles are obviously reflected in the frequency of pulses transmitted through the shear surface. In the presence of transverse mass transfer, the probable trajectories of the particle movement in the rapid shear flow of the uniform granular material will fundamentally differ from the ideal rectilinear trajectories of the translational movement of particles. The additional transverse impulses arising as a result of transverse mass transfer will lead to the generation of an additional component of stresses in the shear flow.

This article discusses the results of theoretical studies of the generation mechanisms of kinetic shear stresses in the rapid gravity flow of cohesionless inelastic spherical particles caused by impulse transfer through the shear surface.

2. Microstructural methods for describing shear stresses in the rapid shear flow of the granular medium

Taking into account the characteristic features of rapid gravity flows, the authors of [10, 15, 16] proposed a defining relation for the component of the stress tensor depending on the shear rate using the methods of statistical physics (mechanics) in relation to the shear flow of smooth inelastic spherical particles of uniform size. According to the proposed relation, the stress is determined taking into account the size, density, and restitution coefficient in binary collisions of particles. In this case, the function of the anisotropic distribution of particles in collision, known from the results of numerical modeling, is used, the only argument of which is the volume fraction of the solid particles. The influence of particle fluctuations on the stress generation is carried out according to the specific kinetic energy of fluctuations (temperature of the granular medium) and its dissipation during collisions. As a result, the dependence of stresses on the shear rate is expressed in the constitutive relation in an implicit form, since it is reflected in the form of the dependence of the specific values of the kinetic energy of fluctuations and its dissipation on the gradients of the granular medium flow velocity.

A detailed analysis of the predictive properties of the above microstructural model of stress generation was carried out in [3] by the method of computer simulation. Despite the obvious advantages of the model consisting in its simplicity and the possibility of taking into account the relationship between the structural, kinematic and dynamic characteristics of the flow without using empirical fitting coefficients, the authors of the work note its limited predictive potential. Problems arise when using the theory to describe the complex profiles of flow characteristics observed in practice in gravity flows, for example, the volume fraction of the solid phase with its maximum values in the central part of the flow. In addition, the proposed constitutive relations for expressing the stress tensor are characterized by insufficiently high predictive power in relation to the formulation of the boundary condition at the interface between the flow and the rough chute. At the same time, an extremely high sensitivity of the calculated values of the flow parameters to the formulation of the boundary conditions is noted.

Adequate modeling requires experimental determination of the distribution function of the solid phase fraction in the near-boundary flow zone for each specific version of the boundary conditions formulation, which is an extremely difficult task. In this regard, there is an urgent need for a theoretical model that predicts the structural characteristics of the gravity flow in its area bordering on a rough bottom.

In addition to the noted shortcomings of the microstructural theory [10], one should also point out the limited consideration of the particle properties (smooth particles) and, largely due to this, the lack of attention to the effects of their interaction, as a result of which particles acquire components of fluctuations leading to their rotation and transverse mass transfer. It is obvious that the flow of the corresponding constituent impulse during the interaction of particles in the process of shear deformation of the granular medium under certain conditions can have effects that significantly determine the dynamics of the shear flow.

The authors of [18] carried out a statistical analysis of the particle interaction under rapid shear deformation and came to the conclusion that the interaction should be accompanied not only by chaotic fluctuations, but also by transverse mass transfer of particles. The most important characteristic of the dynamic interaction of particles participating in the impulse transfer through the shear surface in the process of rapid shear deformation is the collision angle which determines the relationship between the shear and normal components of the transferred impulse. Taking this into account, the authors make a logical conclusion that transverse mass transfer should lead to an increase in the uniformity of the distribution of contact points over the surface of interacting particles. Assuming the decisive role of transverse mass transfer in the mechanism of stress generation, they come to the conclusion about the distribution of contact points in collisions associated with the impulse transfer through the shear surface within the solid angle corresponding to half of the sphere. The consequence of this conclusion is the equality of the total frequency of collisions of particles to the doubled value of the frequency of their collisions, which generate stresses on the shear surface. More important in relation to the flow dynamics is the absence of a dependence of the collision angle of particles on the relative volume fraction of the solid particles.

The certainty introduced in this way into the conditions of contact of particles allowed the authors to obtain the constitutive relations for calculating the shear stresses during the rapid shear flow of the granular medium. In this case, the authors used the Routh tangent impact hypothesis for cohesionless nonelastic spherical particles and expressed stresses based on their relationship with the energy flows generated by shear deformation, kinetic energy of fluctuations, and viscous and elastic dissipation energy. For a two-dimensional steady-state rapid gravity flow in the x direction in the absence of turbulent pulsations, the corresponding energy balance equation is presented in the following form

$$\tau_{yx}^{\text{col}}(du_x / dy) = \gamma_{\text{col}} = NFD_{\text{col}}, \qquad (2)$$

where γ_{col} is specific energy dissipation during fluctuations and collisions of particles per unit volume of the layer; N is the number of particles per unit volume of the layer; D_{col} is specific dissipation of the kinetic energy of the particle per collision; F is particle collision frequency.

When calculating the kinetic energy of particle fluctuations and dissipation, it was assumed that their rotational energy is negligibly small compared to the energy of relative shear displacements and fluctuations [18]. Specific energy dissipation per one collision of a cohesionless spherical particle, which is due to the viscous friction in an interstitial medium and elastic collisions, is expressed in the following form

$$D_{\text{col}} = \rho\pi d^3 / 12(V')^2 \left(1.5C_D s\rho_f / (\rho d) + e\right), \qquad (3)$$

where d is the particle diameter; s is the average distance between particles; C_D is the coefficient of hydraulic resistance equal to 1 under conditions of rapid shear flow; ρ, ρ_f is the density of particles and interstitial medium, respectively; V' is average velocity of particle fluctuations; e is the fraction of the kinetic energy of the particle which is dissipated due to elastic dissipation upon impact.

The fraction of energy dissipation due to elasticity and friction effects is calculated in accordance with the Routh tangent impact hypothesis as a function of the recovery coefficients k and external surface friction μ in the collision of particles [18]

$$e = \left(1-k^2\right)/4 + \mu(1+k)/\pi + \mu^2(1+k)^2/4 . \qquad (4a)$$

However, the research carried out in the framework of [19] indicate that the hypothesis of the Routh tangent impact in cases of particle collision at collision angles close to frontal impact is not correctly used. As a result of the study, it was found that in this case it is more reasonable to use the hypothesis of Newton oblique impact which uses the coefficient of its reduction λ to predict the post-impact value of the tangential velocity component. In this regard, a combined tangent impact hypothesis is proposed in [19] combining the advantages of the Routh and Newton hypotheses according to which the fraction of kinetic energy dissipation due to elasticity and friction effects is calculated as

$$e = \left(1-k^2\right) + \frac{1}{2}\lambda - \frac{1}{8}\mu^2(1+k)^2 - \\ -\frac{1}{8}\lambda^2 + \frac{2}{\pi}\mu(1+k) - \frac{2}{3\pi}\mu\lambda(1+k). \qquad (4b)$$

When calculating the shear stresses τ_{yx}, the authors of [18] used dependence (1) according to which stresses are generated exclusively due to the transfer through the shear surface of the tangential component of the impulse ΔM_x. The magnitude of this impulse component is determined as its average integral value with an isotropic distribution of contact points on the surface of colliding particles within the solid angle corresponding to half of the sphere as a function of the shear rate and the average distance between the centers of particles in contact through the shear surface.

The certainty in the conditions of particle contact, introduced by the authors as a result of assumptions, allowed them to write the constitutive relations between the velocity of fluctuations V', the frequency of particle collisions across the shear surface (generating a shear stress) f, and the total collision frequency F in the following form:

$$V' = F ; \qquad (5a)$$

$$F = 2f. \qquad (5b)$$

Using the obtained relations (1)–(5), the authors of [18] expressed the velocity of fluctuations and the frequency of particle collisions, as well as formulated an expression for calculating the shear stress caused by the transfer of tangential impulse by colliding particles through the shear surface

$$\tau_{yx}^{\mathrm{col}} = v_0 \rho d^2 \left(\frac{\partial u_x}{\partial y}\right)^2 \frac{d}{2s} \times \\ \times \left(\frac{(1+k)^3\left[(0.125+0.144\mu)\beta + (0.053+0.81\mu)\right]^3 \dfrac{d/s}{1+d/s}}{D_{\mathrm{col}}} \right)^{1/2}, \qquad (6)$$

where v_0 is the volume fraction of the solid phase at the densest packing of particles; β is the coefficient that depends on the elastic properties of particles and their volume fraction in the flow (for a developed rapid shear flow $\beta \approx 1$ [10, 15]).

However, it should be noted that this remarkable result became possible due to the assumption of invariant conditions of impact contact interactions of particles. Assuming the decisive role of transverse mass transfer in the formation of conditions for the interaction of particles, the authors of [18] considered it possible to admit an isotropic distribution of the collision angles of particles in the entire range of its maximum possible values, regardless of the volume fraction of the solid particles and the shear rate in the flow. This assumption contradicts the results of studies [20, 21] carried out by the method of computer simulation which reveal an anisotropic (close to normal) distribution of particle collision angles in a rapid shear flow with the presence of some of their characteristic values with their explicit dependence on the volume fraction of the solid particles.

Hereinafter, we mean the angle θ between the shear direction and the normal to the particle surface at the point of contact as the particle collision angle. In this case, contact interactions are taken into account, where the impact impulse transfers a certain momentum in the shear direction $\Delta M_x^{\mathrm{col}}$.

The analysis carried out in the work [22], under the assumption of the normal distribution of the collision angles of uniform spherical particles interacting through the shear surface, made it possible to obtain an expression for calculating the most probable value of their collision angle

$$\theta = \pi - 0.5\arccos\left[\frac{d+s}{\sqrt{3}d}\right] - 0.5\arccos\left[\frac{2s}{\sqrt{3}(d+s)}\right]. \quad (7)$$

The analysis shows that for any volume fraction of the solid particles, the minimum value of the particle collision angle is equal to $\pi/2$, and its maximum possible value depends on the volume fraction of the solid particles and does not exceed π. In accordance with expression (7) and the results of computer simulation carried out with transverse mass transfer of uniform particles [20], the characteristic (most probable) value of the collision angle increases with a decrease in the linear concentration of particles d/s (volume fraction of the solid phase). In this case, the angle value varies from 115° for cases of high concentration of spherical particles, and to 135° for cases of high volume fraction of voids corresponding to linear concentration of $d/s \le 1.85$, which is identical to the volume fraction values of the solid particles $v \le 0.25$.

Thus, in the range of values of the solid phase concentration, characteristic of the rapid gravity flow of granular materials ($v \le 0.5$), there is a significant change in the particle collision angle during the impulse transfer through the shear surface. Consequently, the hypothesis of an invariant isotropic distribution of contact interactions over the particle surface adopted in [18] is completely correct only for sufficiently small values of the volume fraction of the solid particles $v \le 0.25$.

In connection with the above, the relationship between the frequencies f and F included in the constitutive relations (1) and (2) established by the authors in [18] in the form of dependence (5*b*) is fulfilled only at low values of the solid phase concentration ($v \le 0.25$). It can be assumed that at high volume fraction of the solid phase ($v \succ 0.25$), the frequency ratio f/F will be identical to the ratio of the particle surface area, on which contact interactions between particles are accompanied by the generation of shear stresses, to the total particle surface area. Based on this hypothesis, the relationship between the indicated frequencies can be represented as follows

$$f = F(1-\sin\theta')/2, \quad (8)$$

where θ' is the maximum possible value of the particle collision angle participating in the transfer of tangential impulse at a certain concentration of the solid particles (*d/s*). In accordance with expression (7), the dependence for θ' determination can be written in the following formulation

$$\theta' = \arccos\left[\frac{d+s}{\sqrt{3}d}\right] + \arccos\left[\frac{2s}{\sqrt{3}(d+s)}\right]\frac{\pi}{2}. \quad (9)$$

The average integral value of the tangential component of the impulse caused by the action of friction forces and shock impulses over the surface of the contacting particles is calculated depending on the value of the characteristic angle of particle collision (7), which is a function of the volume fraction of the solid phase [22],

$$\Delta M_x^{\text{col}} = S_0 \times \times\left[\frac{\pi}{2}\cos^2\theta + \mu\left(\cos\theta + \frac{1}{2}\sin^2\theta\ln\left(\frac{1+\cos\theta}{1-\cos\theta}\right)\right)\right], \quad (10)$$

where $\theta = \theta^* - \pi/2$; $S_0 = \pi d^3/6\rho(1+k)bd(\partial u_x/\partial y)$ is the impulse of a frontal impact for uniform spherical particles, due to the relative shear component of their rate $bd(\partial u_x/\partial y)$; $b = (\pi/(6v))^{1/3}$ is a geometric parameter which is a function of the volume fraction of the solid particles v.

Taking into account relations (1), (2), (5*a*), (8), (10), the frequency of particle collisions participating in the transfer of tangential impulse through the shear surface can be determined as the following dependence on the shear rate and volume fraction of the solid particles

$$f = \left[\frac{1-\sin\theta'}{2}(1+k)\left[\frac{\pi}{2}\cos^2\theta + \mu\left(\cos\theta + \frac{1}{2}\sin^2\theta \times \times\ln\left(\frac{1+\cos\theta}{1-\cos\theta}\right)\right)\right]\Big/D_{\text{col}}\right]^{1/2}\frac{bd(\partial u_x/\partial y)}{s}. \quad (11)$$

For known values of the collision frequency (11), the average integral value of the tangential component of the impact impulse (10) and the number of particles per unit area of the shear surface, the normal to which has the *y*-direction: $p_y = (bd)^{-2}$, the shear stresses are determined in accordance with the constitutive relation (1) by multiplying the mentioned values.

$$\tau_{yx}^{\text{col}} = (bd)^{-2} f\Delta M_x^{\text{col}}. \quad (12)$$

Thus, both in the previous [18] and in the last [22] formulation of constitutive relations for shear stresses, the hypothesis put forward by Bagnold [17] about the proportionality of the tangential component stresses of the impact impulse transmitted through the shear surface is taken as the basic theory.

The fundamental difference between these options lies in the fact that in the previous version the authors derived relations under the assumption of the decisive role of transverse mass transfer of particles, as a result of which it became possible to assume an isotropic distribution of contact points over their surface. This position contradicts the results of studies [20, 23], which indicate the presence of the characteristic value of the particle collision angle generating shear stresses and its significant dependence on the volume fraction of the solid particles. This is a serious disadvantage of this version, which complicates an adequate assessment of the influence of the structural characteristics of the flow on the dynamic angle of internal friction (the ratio of shear and normal stresses).

In the latter version [22], the constitutive relations take into account the anisotropic distribution of contact points on the surface of interacting particles and allow determining the characteristic values of the collision angle depending on the structural characteristics of the flow. The most fundamental difference of this version from the previous one is the complete disregard for the influence of transverse mass transfer on the stress generation mechanism, which, obviously, is its serious drawback.

3. Results and discussion

The contribution of transverse mass transfer to generating shear stresses in the rapid gravity flow of incoherent nonelastic nonsmooth spherical particles can be estimated based on the formal analogy of the rapid shear flow of particles with a dense gas. The existence of the analogy is confirmed by numerous studies [1, 2, 4, 11, 12–14, 24], which made it possible to identify a granular medium in a state of rapid shear deformation as a “gas of solid particles” and successfully apply a well-developed molecular kinetic theory to describe its states and observed physical effects [4].

In terms of the problem of identifying shear stresses, the most important physical phenomena taking place in a dense gas are viscosity, diffusion, and permeability. These phenomena are important to the extent that each of them reflects the tendency of the medium to transfer its individual elements in a direction transverse to the direction of shear deformation. The higher the permeability of the medium, the more intense the self-diffusion flow and the higher the contribution of the transverse impulses of particles moving between adjacent elementary layers of the shear flow to the change in the longitudinal component of the layer-by-layer impulse. In accordance with the molecular kinetic theory, the permeability of a medium is characterized by the mean free path of particles. According to the research results presented in [19, 25, 26], under conditions of the rapid gravity flow of granular materials, the mean free path of particles can be taken equal to the average distance between particles *s*. This is due to the dominance of the relative shear velocity of particles over the average rate of their fluctuations ($bd(\partial u_x/\partial y) \succ\succ V'$).

Another parameter that directly determines the tendency of the medium to self-penetration is the rate of fluctuations of its elementary particles. The physical quantity proportional to the free path of particles and the average rate of their fluctuations is called the diffusion coefficient, which, in the case of a uniform granular medium, characterizes the intensity of balanced counter flows of chaotically moving particles (quasi-diffusion mixing). Thus, the diffusion coefficient can formally be considered as an abstract characteristic of the medium permeability [27] which determines the intensity of transverse mass transfer.

As a result of transverse mass transfer in the granular medium in the shear flow state, a viscous effect, the magnitude of which will be proportional to the medium permeability and the impulse associated with the shear direction, will arise. Since the magnitude of the impulse is proportional to the shear rate and bulk density of the medium, and the change in the impulse is equal to the impulse of the force, the shear stresses in the granular medium due to transverse mass transfer will be determined in accordance with the molecular kinetic theory as the product of permeability and bulk density and shear rate. For the case of a two-dimensional steady-state shear flow, the transverse mass transfer is limited by counter fluxes of particles in the direction normal to the flat shear surface, which is taken into account in the constitutive relation for calculating the shear stress component by the corresponding proportionality coefficient 1/3

$$\tau_{yx}^{\mathrm{tr}} = 1/3\nu\rho s V'(\partial u_x/\partial y) = \mu_{\mathrm{tr}}(\partial u_x/\partial y), \qquad (13)$$

where μ_{tr} is the coefficient of viscous friction due to transverse mass transfer during shear deformation of the granular medium.

Thus, in accordance with the proposed approach, shear stresses in the rapid gravity flow of the granular material will be formed as a result of the conjugation of their two components. One of the components of the dynamic stress, which depends on the shear rate, will be generated as a result of the exchange of tangential impact impulses between the particles interacting through the shear surface τ_{yx}^{col}. Another component is due to a change in the longitudinal impulse due to transverse mass transfer in the shear flow of particles accumulating different momentum τ_{yx}^{tr}. As a result, shear stresses will be determined as the following sum

$$\tau_{yx} = \tau_{yx}^{\text{col}} + \tau_{yx}^{\text{tr}}. \quad (14)$$

Obviously, to initiate the flow of quasi-diffusion mixing and, as a consequence, transverse mass transfer, a constant supply of energy is required, which is provided due to the gravitational shear of the granular medium on a rough chute. The intensity of the corresponding energy flux is proportional to the shear deformation rate and the tangential stress component τ_{yx}^{tr} generated by transverse mass transfer

$$A_{\text{tr}} = \tau_{yx}^{\text{tr}} \left(\partial u_x / \partial y\right). \quad (15)$$

This energy is spent on overcoming the resistance of the granular medium to its transverse penetration by individual particles moving from one elementary layer of the shear flow to another. The viscous resistance to the movement of a sphere in the granular medium in accordance with the molecular kinetic theory is expressed by the following relationship [28]

$$P_{\text{tr}} = 3\pi d\, \mu_{\text{tr}}\, V_{\text{tr}}, \quad (16)$$

where V_{tr} is an averaged component of the particle rate during transverse mass transfer, which, in the first approximation, can be taken equal to the averaged particle fluctuation velocity.

In this case, the energy dissipation flux per unit volume of the granular medium due to pseudo-viscous effects in transverse mass transfer of particles, neglecting the effect of viscous dissipation in the interstitial medium, will be defined as

$$\gamma_{\text{tr}} = NP_{\text{tr}} V_{\text{tr}} = N\pi d \nu \rho s V' V_{\text{tr}}^2. \quad (17)$$

It is obvious that the value of the total energy dissipation flux during the rapid shear flow of the granular medium can be expressed by summing the fluxes of energy dissipation in inelastic collisions of nonsmooth particles, viscous dissipation during their fluctuations in an interstitial medium, and penetration of the shear flow during transverse mass transfer. The total flux of the mentioned components of energy dissipation can be represented as the sum of fluxes (2) and (17)

$$\gamma = \gamma_{\text{col}} + \gamma_{\text{tr}} = NFD_{\text{col}} + N\pi d\, \nu \rho\, s V' V_{\text{tr}}^2. \quad (18)$$

Before proceeding with the formulation of the equations for impulse and energy conservation, we should introduce the concept of the temperature of the granular medium (granular temperature) E which means the total kinetic energy of the main forms of mutual displacements of particles per unit volume of the flow. Neglecting the energy of the proper rotations of particles, which is quite acceptable for the rapid shear flow of cohesionless granular materials [29], we express the temperature of the granular medium as

$$E = E_{\text{sh}} + E_{\text{fl}} + E_{\text{tr}}, \quad (19)$$

where $E_{\text{sh}}, E_{\text{fl}}, E_{\text{tr}}$ are components of the kinetic energy of particles due to the presence of the relative shear velocity, fluctuations and transverse mass transfer, respectively.

The impulse conservation equation is obtained by analyzing the dynamic equilibrium of an elementary volume of the granular medium arbitrarily selected in the flow. For a two-dimensional rapid gravity flow, taking into account inertial forces, shear stresses and gravity action, the following form can be obtained

$$\nu\rho\left(\frac{\partial u}{\partial \tau} + u_x \frac{\partial u_x}{\partial x} + u_y \frac{\partial u_y}{\partial y}\right) = \mu_{\text{dyn}} \frac{\partial p}{\partial y} + \nu \rho g \sin\alpha, \quad (20)$$

where p is lithostatic pressure; μ_{dyn} is an angle of internal dynamic friction equal to the ratio of shear stress to normal, from which it follows: $\mu_{\text{dyn}} p = \tau_{yx} = \tau_{yx}^{\text{c}} + \tau_{yx}^{\text{col}} + \tau_{yx}^{\text{tr}}$; τ_{yx}^{c} are contact shear stresses independent of the shear rate [10]; α is an angle of inclination of the rough chute to the horizon.

The energy conservation equation determines the balance of energy fluxes in relation to the elementary volume of the rapid gravity flow of the granular medium. Taking into account energy fluxes associated with a change in the temperature of the granular medium, the energy generated by the shear and energy dissipation due to the effects of elasticity, friction, viscosity of the interstitial medium and transverse mass transfer, the following form can be obtained for a two-dimensional gravity flow

$$\frac{\partial E}{\partial \tau} + u_x \frac{\partial E}{\partial x} + u_y \frac{\partial E}{\partial y} = \mu_{\text{dyn}} p \frac{\partial u_x}{\partial y} - \gamma_{\text{col}} - \gamma_{\text{tr}}. \quad (21)$$

When formulating the problems of the dynamics of rapid gravity flows, a special problem consists in the adequate determination of the conditions at the rough bottom and at the open surface of the flow, i.e. at $y = 0, h$, respectively. As for the boundary condition at the open surface of the flow, in this case the conditions that determine the absence of mass, impulse and energy fluxes through the mentioned boundary seem to be quite adequate [3]

$$\nu\rho\frac{\partial u_y}{\partial y} = \nu\rho u_y\frac{\partial u_y}{\partial y} = u_y\frac{\partial E}{\partial y} = 0. \qquad (22)$$

Another problem is the formulation of the boundary condition at the bottom of the flow with a rough base. It has been found that this condition fundamentally affects the dynamics of the velocity fields and the volume fraction of the solid particles in the gravity flow [3]. In order to achieve the required accuracy of its formulation, it is advisable to follow principles similar to those used to derive the equations of flow dynamics. Assuming the size of the bottom roughness and their shape to be identical to half of the volume of spherical particles, we can accept zero conditions for the sliding rate, i.e. at $y = 0$, $u_x = 0$.

However, in accordance with the principles used to derive the equations for impulse and energy conservation, the boundary condition must reflect the specificity of shear stresses generation in the boundary zone immediately adjacent to the surface of the rough chute within the layer thickness equal to bd. The specific conditions for generating stresses in this zone are related to the fact that their pseudo-viscous component, due to transverse mass transfer τ_{yx}^{tr}, manifests itself as a result of a predominantly one-way exchange of particles with the upper part of the flow. Such conditions will cause a two-fold decrease in the pseudo-viscous component of the stress in the mentioned boundary flow zone, i.e.:

$$\tau_{yx}^{\mathrm{tr}} = 1/6\nu\rho s V'\frac{\partial u_x}{\partial y}, \text{ at } 0 \le y \le bd\,. \qquad (23)$$

Obviously, under this condition, dynamic equilibrium in the flow will be achieved in the boundary zone due to an increase in the shear rate and a decrease in the volume fraction of the solid phase, which is confirmed by the results of experimental studies [2, 3, 25, 26].

4. Conclusion

A mathematical model of kinetic shear stresses in the rapid shear gravity flow of the uniform granular medium is proposed, according to which stresses are generated due to the transfer of tangential impulses through the shear surface by two different mechanisms. One of the mechanisms provides the impulse transfer due to the impact interactions of particles, taking into account the conditions of their contact which depend on the volume fraction of the solid particles. In accordance with the second mechanism, the tangential impulse is transmitted as a result of the transverse quasi-diffusion transfer of particles. The mathematical model describing the mentioned mechanisms makes it possible to determine the magnitude of kinetic shear stresses in the rapid gravity flow of uniform cohesionless spherical particles depending on their properties, as well as structural and kinematic parameters of the flow. As part of further research, it is planned to test the prognostic properties of the proposed mathematical model using computer and physical modeling methods.

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Conflict of interests

The authors declare no conflict of interest.

References

1. Nagel SR. Experimental soft-matter science. *Reviews of modern physics.* 2017;89(2):025002. DOI:10.1103/RevModPhys.89.025002

2. Dolgunin VN, Kudi AN, Tuev MA. Mechanisms and kinetics of gravity separation of granular materials. *Physics-Uspekhi.* 2020;63(6):545-561. DOI:10.3367/UFNe.2020.01.038729

3. Hutter K, Sheiwiller T. Rapid plane flow of granular materials down a chute. In: Jenkins JT, Satake M. (eds.) *Studies in Applied Mechanics*. 1983;7:283-293. DOI:10.1016/B978-0-444-42192-0.50029-3

4. Brennen CE. *Fundamentals of Multiphase Flows.* Cambridge: Cambridge University Press; 2005. 410 p. DOI:10.1017/CBO9780511807169

5. Domnik B, Pudasaini SP, Katzenbach R, Miller SA. Coupling of full two-dimensional and depth-averaged models for granular flows. *Journal of Non-Newtonian Fluid Mechanics.* 2013;201:56-68. DOI:10.1016/j.jnnfm.2013.07.005

6. Hill KM, Fan Y. Granular temperature and segregation in dense sheared particulate mixtures. *KONA Powder and Particle Journal*. 2016;33:150-168. DOI:10.14356/kona.2016022

7. Gray J, Shearer M, Thornton AR. Time-dependent solutions for particle-size segregation in shallow granular avalanches. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences.* 2006;462:947-972. DOI:10.1098/rspa.2005.1580

8. Jesuthasan N, Baliga BR, Savage SB. Use of particle tracking velocimetry for measurements of granular flows: Review and application – Particle tracking velocimetry for granular flow measurements. *KONA Powder and Particle Journal*. 2006;24:15-26. DOI:10.14356/kona.2006006

9. Dolgunin VN, Kudi AN, Ivanov OO, Tuev MA. Interrelation of structural and kinematic characteristics during free-surface gravity flows of granular materials. *Proceedings of the Seventh International Conference on Advances in Civil, Structural and Mechanical Engineering – CSM 2018, 27-28 October 2018, Rome, Italy.* New York: Institute of Research Engineers and Doctors; 2018. p. 6-12. DOI:10.15224/978-1-63248-163-4-14

10. Savage SB. Granular flows down rough inclines - Review and extension. In: Jenkins JT, Satake M. (eds.) *Studies in Applied Mechanics*. 1983;7:261-282. DOI:10.1016/B978-0-444-42192-0.50028-1

11. Forterre Y, Pouliquen O. Flows of dense granular media. *Annual Review of Fluid Mechanics.* 2008;40:1-24. DOI:10.1146/annurev.fluid.40.111406.102142

12. Pudasaini SP. A general two-phase debris flow model. *Journal of Geophysical Research: Earth Surface*. 2012;117(3):F03010. DOI:10.1029/2011JF002186

13. Goldhirsch I. Rapid granular flows. *Annual Review of Fluid Mechanics*. 2003;35:267-293. DOI:10.1146/annurev.fluid.35.101101.161114

14. Gidaspow D. *Multiphase flow and fluidization: Continuum and kinetic theory description.* San Diego: Academic press; 1994. 467 p.

15. Bagnold RA. Experiments on a gravity free dispersion of large solid spheres in a newtonian fluid under shear. *Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences*. 1954;225(1160):49-64. DOI:10.1098/rspa.1954.0186

16. Savage SB, Jeffrey DJ. The stress tensor in a granular flow at high shear rates. *Journal of Fluid Mechanics*. 1981;110:255-272. DOI:10.1017/S0022112081000736

17. Jenkins JT, Savage SB. A theory for the rapid flow of identical, smooth, nearly elastic, spherical particles. *Journal of Fluid Mechanics*. 1983;130:187-202. DOI:10.1017/S0022112083001044

18. Ackermann NL, Shen HH. Rapid shear flow of densely packed granular solids. In: Jenkins JT, Satake M. (eds.) *Studies in Applied Mechanics*. 1983;7:295-304. DOI:10.1016/B978-0-444-42192-0.50030-X

19. Dolgunin VN, Ivanov OO, Ukolov AA. Segregation kinetics of particles with different roughness and elasticity under a rapid gravity flow of a granular medium. *Theoretical Foundations of Chemical Engineering.* 2009;43(2):187-195. DOI:10.1134/S0040579509020092

20. Campbell CS, Brennen CE. Computer simulation of shear flows of granular materials. In: Jenkins JT, Satake M. (eds.) *Studies in Applied Mechanics*. 1983;7:313-326. DOI:10.1016/B978-0-444-42192-0.50032-3

21. Windows-Yule C, Tunuguntla D, Parker D. Numerical modeling of granular flows: a reality check. *Computational Particle Mechanics.* 2016;3(3):311-332. DOI:10.1007/s40571-015-0083-2

22. Dolgunin VN, Ukolov AA, Klassen PV. Model of segregation mechanism at quick shear flow of particles. *Teoreticheskiye osnovy khimicheskoy tekhnologii = Theoretical Foundations of Chemical Engineering*. 1992;26(5):707-716. (In Russ.)

23. Campbell CS, Brennen CE. Chute flows of granular material: Some computer simulations. *Journal of Applied Mechanics*. 1985;52(1):172-178. DOI:10.1115/1.3168990

24. Jenkins J, Berzi D. Kinetic theory applied to inclined flows. *Granular Matter.* 2012;14(2):79-84. DOI:10.1007/s10035-011-0308-x

25. Dolgunin V, Kudy A, Ukolov A, Development of the model of segregation of particles undergoing granular flow down on inclined chute. *Powder Technology.* 1998;96(3):211-218. DOI:10.1016/S0032-5910(97)03376-7

26. Dolgunin V, Kudi A, Ukolov A, Tuev M. Rapid granular flows on a vibrated rough chute: Behavior patterns and interaction effects of particles. *Chemical Engineering Research and Design.* 2017;122:22-32. DOI:10.1016/j.cherd.2017.03.038

27. Chuyev AS. On the inconsistency of definitions of physical quantities dynamic and kinematic viscosity. *Zakonodatel'naya i prikladnaya metrologiya.* 2012;1:54-60. (In Russ.)

28. Ferziger JH, Kaper HG. *Mathematical theory of transport processes in gases*. Amsterdam: North Holland Publishing; 1972. 568 p.

29. Rahaman MF, Naser J. Kinetic theory of granular flow: Effect of rotary movement on solids pressure and viscosity. *Proceedings of the 4-th World Congress on Particle Technology, 21-25 July 2002, Sydney, Australia.* [Full text of paper in CD-ROM].

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