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Mathematical modeling of the vacuum extraction process in a capillary-porous body – liquid system

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Abstract. This article examines the topical issue of mathematical modeling of plant material extraction processes, where a deep understanding and analysis of these processes are important in various industries, including food, pharmaceutical, and chemical. It has been established that the mathematical model under study is a special case of a more general universal model describing heat and mass transfer in a capillary-porous body-liquid system. The solution to the boundary value problem obtained in the work using exponential Fourier series averages based on a continuous periodic function reveals the properties and approximation capabilities of this model. The theorem on the convergence of the model to the physical system describing the processes of plant material extraction under certain conditions confirms its reliability and applicability. The issues of summability of trigonometric series in the metric of the corresponding functional space are analyzed. The study provides a fundamental scientific description of the mathematical model of plant material extraction processes and a detailed analysis of its properties, opening new horizons for the development of this field of research. The presented results will be useful for specialists in fields related to extraction, as well as for those involved in the development of new methods for optimizing and controlling plant material extraction processes.

Keywords: capillary-porous materials; extraction; mathematical model; Fourier series.

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Математическое моделирование процесса экстрагирования в системе капиллярно-пористое тело – жидкость

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Аннотация. Исследуется актуальная проблема математического моделирования процессов экстракции растительного материала, где глубокое понимание и анализ данных процессов имеют важное значение в различных отраслях промышленности, включая пищевую, фармацевтическую и химическую. Установлено, что изучаемая математическая модель представляет собой частный случай более общей универсальной модели, описывающей тепломассоперенос в системе капиллярно-пористое тело – жидкость. Решение краевой задачи, полученное в работе с использованием экспоненциальных средних ряда Фурье, основанных на непрерывной периодической функции, выявляет свойства и аппроксимационные возможности этой модели. Теорема о сходимости модели к физической системе, описывающей процессы экстракции растительного материала, при определенных условиях подтверждает ее достоверность и применимость. Проанализированы вопросы суммируемости тригонометрических рядов в метрике соответствующего функционального пространства. Проведенное исследование предоставляет фундаментальное научное описание математической модели процессов экстракции растительного материала и подробный анализ ее свойств, открывая новые горизонты для развития данной области исследований. Представленные результаты будут полезны для специалистов в областях, связанных с экстракцией, а также для тех, кто занимается разработкой новых методов оптимизации и контроля процессов экстракции растительного материала.

Ключевые слова: капиллярно-пористые материалы; экстрагирование; математическая модель; ряды Фурье.

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1. Introduction

Over the past decades, the extraction of plant raw materials has become one of the most promising and actively researched areas in the chemical and biochemical industries. This is due to the growing interest in natural ingredients, which are widely used in various industries, including the food, cosmetics, pharmaceutical, and fragrance industries.

Biologically active substances isolated during the extraction of plant raw materials, such as flavonoids, terpenoids, phenols, and others, are characterized by a variety of pharmacological and biological properties that are used to create new medicines, functional foods, aromatic and cosmetic products.

To extract biologically active substances from plant components, it is necessary to carry out effective and environmentally safe extraction processes. Traditional extraction methods (percolation, maceration, and extraction using organic solvents) have a number of disadvantages, including:

– a low extraction rate; long time required to obtain a high-quality extract, which makes it difficult to scale up the process and reduces its efficiency;

– the use of high temperatures, which can lead to the destruction of thermolabile components being extracted;

– unevenness and incompleteness of the extraction of active components from plant material;

– an increased risk of product contamination with residual solvents.

These disadvantages have sparked researchers' interest in developing new and improved extraction methods that are highly efficient, economical, and environmentally friendly. Among them, it is worth noting the use of high pressures and temperatures, ultrasonic extraction, microwave extraction, extraction using supercritical solvents, and other processes that have great potential for improving the extraction of plant raw materials.

Mass transfer processes in solid-phase systems (extraction) are widely used across various industries, including chemical, food, textile, pulp and paper, pharmaceutical, and other sectors. These processes are highly resource- and energy-intensive, making improvements to equipment and technological design a pressing issue.

Mathematical modeling of mass transfer processes in solid-phase systems based on solving differential equations of mass and heat transfer is currently being actively developed around the world. Mathematical methods for describing kinetics (analytical and numerical) are now widely used not only for the kinetic calculation of equipment, but also for computer modeling of processes in order to select the correct equipment and technological design.

Research in the field of modeling plant raw material extraction processes is becoming increasingly important, as it allows the optimization of process parameters and conditions to achieve maximum efficiency and reduce costs. Modeling can be used to analyze the influence of various factors, such as solvent type, temperature, pressure, and stirring speed, on the output characteristics of the extraction process.

It is also important to consider the influence of the micro- and macrostructure of plant material on the extraction process. In this regard modeling mass transfer processes within the material allows for a more accurate description of extraction processes and prediction of their results.

Achievements in the field of modeling the extraction process of plant raw materials include the development of mathematical models based on mass and heat transfer, as well as chemical kinetics. Some of these models take into account mass transfer between phases (solvent and plant materials), as well as the physical and chemical properties of components, including diffusion coefficients, equilibrium concentrations, and extraction reaction rates.

Various materials are used in the modeling of the plant material extraction process, including different types of extractors (e.g., with static or dynamic mixing), special installations, and devices for measuring the physical and chemical properties of components. Modern software packages such as COMSOL, Aspen Plus, ANSYS Fluent, and others are used for numerical calculations. These programs allow modeling complex processes taking into account various physical phenomena such as mass transfer, heat transfer, and multiphase flows.

Among the new modern methods of analytical and numerical solution used in the field of modeling the extraction process of plant raw materials, optimization methods such as genetic algorithms, artificial intelligence methods, neural networks, and multi-frequency modeling can be distinguished. Such methods can significantly improve the accuracy and efficiency of the extraction process modeling.

However, there are still many aspects of the modeling of the plant raw material extraction process that remain unexplored. Of particular interest is the study of the interaction of various plant materials with different solvents. In addition, it is necessary to take into account changes in the properties of plant raw materials during the extraction process, such as changes in the concentration and physicochemical properties of the components.

Current trends in the development of mathematical modeling of plant raw material extraction processes are reflected in numerous scientific studies by domestic and foreign authors [1–18].

In this regard, the use of Fourier series is becoming a relevant and promising approach to the analysis and modeling of plant raw material extraction processes.

Fourier series are a mathematical tool that allows (under certain conditions) the periodic function of an information source to be decomposed into an infinite sum of harmonic functions. This approach is widely used in various fields of science and technology, including physics, electrical engineering, signal theory, image and sound processing. In the context of modeling plant raw material extraction processes, Fourier series provide the ability to describe heat and mass transfer in a capillary-porous body-liquid system.

2. Materials and Methods

2.1. Formulation and solution of the boundary value problem

Let us consider one of the possible models of the extraction process. Let there be an extractant in a certain medium filling a hollow straight tube of length *l*, the distribution of which is described by a one-dimensional concentration field, i.e. a function of coordinates and time $u = u(x,t)$. Assume that at any time *t*, the extractant concentration is uniform over the tube cross-section *x*. We consider only the cases with constant diffusion coefficients *D* and porosity *c*. Sources of the substance with density $F_0(x,t)$ may exist inside the tube; denote $F(x,t) = \dfrac{F_0(x,t)}{c}$, $a^2 = \dfrac{D}{c}$. The extraction process under consideration can be formalized as follows [12]: it is required to find the solution

$$u = u(x,t), \quad 0 \le x \le l, \quad t > 0$$

of the following problem

$$u'_t = a^2 u''_{xx} + F(x,t); \tag{2.1}$$

$$u(0,t) = \mu(t); \tag{2.2}$$

$$u'_x(l,t) = 0; \tag{2.3}$$

$$u(x,0) = \varphi(t). \tag{2.4}$$

It is assumed that the functions $\mu = \mu(t)$, $\varphi = \varphi(t)$, $F(x,t)$ are given and Lebesgue-integrable over the corresponding domain of definition.

The physical meaning of constraints (2.2)–(2.4) is as follows.

Condition (2.2) specifies the law of change in the concentration of the extractant at the left end of the tube. Condition (2.3) means that the right end of the tube is impermeable to the extractant. The function $\varphi(x)$ determines the initial distribution of concentrations at the points of the tube.

There may also be a case where the left end of the tube is impermeable, and the law of concentration change is specified at the right end:

$$u(l,t) = \mu(t); \tag{2.2*}$$

$$u'_x(0,t) = 0. \tag{2.3*}$$

In this regard we note the following point. The problem (2.1), (2.2*), (2.3*), (2.4) by replacing the spatial variable: $x = 1 - z$ essentially reduces to the problem (2.1), (2.2), (2.3), (2.4). However, from the point of view of implementing the initial condition (2.20*) see the statements of Theorem 2 below – the problem (2.1), (2.2*), (2.3*), (2.4) is of independent interest.

For functions $\varphi(x)$, that are "sufficiently regular", i.e., represent table by the sum of a Fourier series, the solution of the problem (2.1)–(2.4) using the Fourier method is known: see, for example [12, p. 100–102]. The Fourier method has several advantages compared to other possible approaches, such as the transparency of all steps in the solution, guaranteed convergence of the process (the resulting series) under certain restrictions on $\varphi(x)$, and simplicity of the computational procedure (calculation of finite sums with a sufficiently large number of terms). However, in the general case of Lebesgue-integrable functions $\varphi(x)$, the direct use of Fourier series is not permissible due to known examples of their divergence. Here, we propose to address this issue by considering the so-called exponential averages of Fourier series. The corresponding theory is presented in Section 3.

For the sake of completeness, we briefly present the scheme for solving the problem (2.1)–(2.4), noting the similarity of the mathematical formulation to that given in [18].

In the first step, the change of variables

$$v(x,t) = u(x,t) - \mu(t) \tag{2.5}$$

transform the problem (2.1)–(2.4) to the form

$$v'_t = a^2 v''_{xx} + F(x,t) - \mu'(t);$$

$$v(0,t) = 0;$$

$$v'_x(l,t) = 0.$$

$$v(x,0) = \varphi(x) - \mu(0).$$

Next, let

$$v(x,t) = H(x,t) + J(x,t), \tag{2.6}$$

so that the functions $H = H(x,t)$ and $J = J(x,t)$ are respective solutions of the problem

$$\begin{cases} H'_t = a^2 H''_{xx}; \\ H(0,t) = 0; \\ H'_x(l,t) = 0; \\ H(x,0) = \varphi(x) - \mu(0) \end{cases} \tag{2.7}$$

with homogeneous boundary conditions

$$J'_t = a^2 J''_{xx} + M(x,t); \tag{2.8}$$

$$J(0,t) = 0; \tag{2.9}$$

$$J'_x(l,t) = 0; \tag{2.10}$$

$$J(x,0) = 0, \tag{2.11}$$

where $M(x,t) = F(x,t) - \mu'(t)$.

Problem (2.7) can be solved using the Fourier method [12, pp. 90–94; 13, pp. 154–155]. Specifically, we use individual solutions of partial differential equations and construct a general solution as the sum of individual solutions $P(x,t) = X(x)\,Y(t)$. As can be readily verified, the equation is satisfied by one-parameter families of functions of the form

$$X = X_n(x) = D_n \sin\lambda_n x;$$

$$Y_n(t) = A_n \exp\left(-a^2\lambda_n^2 t\right),$$

where $\lambda_n = \frac{\pi}{l}\left(n - \frac{1}{2}\right)$, $n = 1, 2, \ldots$

Taking into account the boundary conditions, we obtain

$$P_n(x,t) = b_n \exp\left(-a^2\lambda_n^2 t\right)\sin\lambda_n x.$$

The form of the coefficients

$$b_n = A_n D_n,\ n = 1, 2, \ldots,$$

will be determined later on the basis of the initial condition. To implement this condition, we construct an exponential-trigonometric series of the form

$$P(x,t) \sim \sum_{n=1}^{\infty} b_n \exp\left(-a^2\lambda_n^2 t\right)\sin\lambda_n x, \tag{2.12}$$

with the understanding that $P(x,0)$ coincides (in a certain sense specified below) with the function $f(x) = \varphi(x) - \mu(0)$.

It can be easily verified (see, e.g., [12, pp. 52–57]) that the system of functions

$$\left\{\sqrt{\frac{l}{2}}\sin\lambda_n x\right\},\quad n = 1, 2, \ldots, \tag{2.13}$$

is orthonormalized on the interval (0, *l*).

Hence, the formal trigonometric expansion

$$\varphi(x) - \mu(0) \sim \sum_{n=1}^{\infty} b_n \sin\lambda_n x, \tag{2.14}$$

can be constructed, where the coefficients b_n in (2.14) are now the Fourier sine coefficients of the function $f(x) = \varphi(x) - \mu(0)$:

$$b_n = b_n(f) = \frac{2}{l}\int_0^l (\varphi(\tau) - \mu(0))\sin\lambda_n\tau\, d\tau,\quad n = 1, 2, \ldots$$

Let us now turn to the problem (2.8)–(2.11). Were present the unknown function $J(x,t)$ and the given function $M(x,t)$ by their expansions in the system $\{\sin\lambda_n x\}$ (see (2.13)).

Let $\rho_n(t)$ and $\delta_n(t)$ be the Fourier coefficients of the following two functions depending on *t*:

$$J(x,t) \sim \sum_{n=1}^{\infty} \rho_n(t)\sin\lambda_n x; \tag{2.15}$$

$$M(x,t) \sim \sum_{n=1}^{\infty} \delta_n(t)\sin\lambda_n x; \tag{2.16}$$

By substituting expansions (2.15) and (2.16) into equation (2.8), we obtain equality of the corresponding coefficients

$$\rho_n'(t)+a^2\lambda_n^2\rho_n(t)=\delta_n(t),\ n=1,2,... \qquad (2.17)$$

Equation (2.17) together with the initial condition (see (2.11) and (2.15)) $\rho_n(0)=0$ constitutes a Cauchy problem, the solution of which, as can be easily verified, is a sequence of the form

$$\rho_n(t)=\exp\left(-a^2\lambda_n^2 t\right)\int_0^t \exp\left(a^2\lambda_n^2\tau\right)\delta_n(\tau)\,d\tau,$$
$$n=1,2,\ldots \qquad (2.18)$$

In view of (2.15) and (2.18), we obtain the expansion

$$J(x,t)\sim\sum_{n=1}^{\infty}\left[\exp\left(-a^2\lambda_n^2 t\right)\int_0^t \exp\left(a^2\lambda_n^2\tau\right)\delta_n(\tau)\,d\tau\right]\sin\lambda_n x, \qquad (2.19)$$

in which (see (2.16))

$$\delta_n(\tau)=\frac{2}{l}\int_0^t M(z,\tau)\sin\lambda_n z\,dz.$$

Combining relations (2.5), (2.6), (2.12), and (2.19), we finally obtain the solution of problem (2.1)–(2.4) in the form of an exponential-trigonometric series sum

$$u(x,t)=\mu(t)+\sum_{n=1}^{\infty}b_n\exp\left(-a^2\lambda_n^2 t\right)\sin\lambda_n x+$$
$$+\sum_{n=1}^{\infty}\left[\exp\left(-a^2\lambda_n^2 t\right)\int_0^t \exp\left(a^2\lambda_n^2\tau\right)\delta_n(\tau)\,d\tau\right]\sin\lambda_n x,$$

whose coefficients λ_n, b_n, $\delta_n(\tau)$ were defined above, respectively, as $\lambda_n=\frac{\pi}{l}\left(n-\frac{1}{2}\right)$,

$$b_n=\frac{2}{l}\int_0^l(\varphi(x)-\mu(0))\sin\lambda_n x\,dx;$$

$$\delta_n(\tau)=\frac{2}{l}\int_0^l(F(z,\tau)-\mu'(\lambda))\sin\lambda_n z\,dz,\ n=1,2,\ldots$$

2.2. Clarification of the initial condition. Convergence (2.12)

In the boundary-value problem formulated above (2.7), the initial condition (as already noted) requires that at $t=0$ the sum of the Fourier series (2.12) coincide with values $f(x)=\varphi(x)-\mu(0)$. However, a formal substitution of $t=0$ into (2.12) does not guarantee such a coincidence, even if the resulting Fourier series converges. Therefore, the initial condition under consideration should be understood in a more general sense:

$$\lim_{t\to+0}\sum_{n=1}^{\infty}b_n(f)\exp\left(-a^2\lambda_n^2 t\right)\sin\lambda_n x=f(x). \qquad (2.20)$$

Requirement (2.20) is then realized as a problem concerning the limiting behavior of the exponential averages of the Fourier series expansions of function f according to the corresponding system of sines; this problem is also of independent interest (see, e.g., [17]). For convenience the function f can be extended as an odd function and then made $4l$-periodic.

Since the function $\varphi(x)$ is Lebesgue integrable, then $\|f\|<\infty$, where

$$\|f\|=\frac{2}{l}\int_0^l|\varphi(x)-\mu(0)|\,dx,$$

Furthermore, the following estimate holds

$$|b_n(f)|\le\|f\|,\ n=1,2,...$$

Then the series

$$\sum_{n=1}^{\infty}\|f\|\exp\left(-\left(\frac{\pi}{l}a\right)^2\left(n-\frac{1}{2}\right)^2 t\right)$$

will serve as a majorant for (2.12). The convergence of this positive series for each $t>0$ is evident, from which the uniform convergence of (2.12) with respect to x follows.

Problem (2.7), in which boundary conditions (2.2) and (2.3) are transformed into the form (2.2*) and (2.3*) respectively, is solved using the same algorithm, but the sine series expansions will be replaced by cosine series expansions [13, p. 182]. Accordingly, the initial condition (2.20) takes the form

$$\lim_{t\to+0}\sum_{n=1}^{\infty}b_n\exp\left(-a^2\lambda_n^2 t\right)\cos\lambda_n x=f(x); \qquad (2.20^*)$$

$$b_n=\frac{2}{l}\int_0^l(\varphi(x)-\mu(0))\cos\lambda_n x\,dx.$$

3. Results and Discussion

3.1. Implementation of the initial condition

Results (2.20) and (2.20*) can be derived as a consequence of solving a more general problem concerning the behavior, as $t\to+0$, of families of means

$$U_t(f)=U_t(f,x;v)=\sum_{k=1}^{\infty} b_k(f)v_k(t)\sin\frac{\pi}{l}\left(k-\frac{1}{2}\right)x;$$

$$\widetilde{U}_t(f)=\widetilde{U}_t(f,x;v)=\sum_{k=1}^{\infty} b_k(f)v_k(t)\cos\frac{\pi}{l}\left(k-\frac{1}{2}\right)x \tag{3.1}$$

of the Fourier series of function f according to the corresponding system of sines/cosines. The following standard conditions are imposed on the summation sequence $\{v_k(t)\}$:

$$v_k(0)=1,\quad k=1, 2, \ldots; \tag{3.2}$$

$$\lim_{k\to+\infty} v_k(t)=0,\quad t>0; \tag{3.3}$$

$$\lim_{k\to+0} v_k(t)=1,\quad k=1, 2, \ldots; \tag{3.4}$$

$$v_k(t)\ln k=O_t(1),\ k=2, 3, \ldots, t>0. \tag{3.5}$$

In particular cases of (2.20), (2.20*) the families (3.1) take the form

$$v_k(t)=\exp\left(-\theta\left(k-\frac{1}{2}\right)^2 t\right),\ k=1, 2, \ldots; \tag{3.6}$$

and for (3.6) conditions (3.2)–(3.5) are satisfied, as can be readily verified.

Let us establish that sequence (3.6) belongs to the class of piecewise convex sequences. This means that the second differences

$$\Delta^2 v_k(t)=\Delta(\Delta v_k(t)),$$

where $\Delta v_k(t)=v_k(t)-v_{k+1}(t)$, either retain their sign or change it a finite number of times.

In case of (3.6) the signs of $\Delta^2 v_k(t)$ are determined by the signs of the second derivative (see [14]) of the function

$$E(y)=\exp\left(-\theta\, t\left(y-\frac{1}{2}\right)^2\right);$$

here $\theta\in(0;1)$ is a constant. Furthermore, the second derivative

$$E''(y)=-\theta\, t\left(2-\theta\, t\,(2y-1)^2\right)\exp\left(-\theta\, t\left(y-\frac{1}{2}\right)^2\right)$$

obviously changes its sign at only a single point $y\geq 1$. Therefore, sequence (3.6) is piecewise convex, as asserted.

Summation of Fourier series of 2π-periodic functions using piecewise convex methods based on the classical orthogonal trigonometric system

$$\left\{\frac{1}{2};\ \cos x, \sin x; \ldots, \cos kx, \sin kx; \ldots\right\}$$

was studied in [14, 15].

In particular, in [15] the exponential means of Fourier series with respect to the classical sine system were applied to the mathematical modeling of drying processes of plant materials.

In the context of implementing the initial condition (2.4), we extend the methods of the aforementioned works to expansions with respect to the systems

$$\left\{\sin\frac{\pi}{l}\left(k-\frac{1}{2}\right)x\right\} \text{ and } \left\{\cos\frac{\pi}{l}\left(k-\frac{1}{2}\right)x\right\},\ k=1, 2, \ldots,$$

the orthogonality of the first system is noted above (see (2.13)), while the orthogonality of the second one (i.e., the cosine system) on $(0;l)$ can be verified directly from the definition; see also [13, p. 85].

Let us denote by L^p the class of functions $f(x)$, that are summable with p-th power $p\geq 1$ on $(0, l)$ and extended to the entire real axis either as odd or even $4l$-periodic functions. Let $\|f\|_p=\left(\frac{1}{l}\int_0^l |f(t)|\,dt\right)^{1/p}$ denote the norm in L^p. We also denote by $C(C>0)$ constants that may depend only on the explicitly specified indices. The following result holds.

Theorem 1. Let the conditions (3.2)–(3.5) be satisfied for a piecewise convex summation sequence $\{v_k(t)\}$. Then

1) the maximal "strong-type" estimate holds

$$\|\sup_{t>0}|U_t(f)|\|_p\leq C_p\,\|f\|_p,\quad p>1;$$

2) in the case $p=1$ the "strong-type" estimate is replaced by the following relation:

$$\sup_{t>0}\|U_t(f)\|_1\leq C\,\|f\|_1;$$

and the "weak type" estimate

$$\operatorname{mes}\left\{x\in(0;l)\,|\,\sup_{t>0}|U_t(f)|\geq\zeta>0\right\}\leq\frac{C}{\zeta}\|f\|_1.$$

3) the relation

$$\lim_{t\to+0} U_t(f,x;v)=f(x) \tag{3.7}$$

holds almost everywhere on (0, l) and in the metric of the corresponding functional space for any odd function $f\in L^p$ ($p\ge 1$). The equality (3.7) also holds at each point of continuity of the function $f(x)$, and in the case of an everywhere continuous function, it holds uniformly in x.

In particular, the requirement (2.20) is satisfied almost everywhere and at each point of continuity of the function $f(x)$.

The statement of Theorem 1, Part 3, and its proof are given in [18]. The corresponding result for cosine series expansions, together with its proof, is given in the following section. In particular, thanks to this result, a positive answer will be obtained regarding the implementation of condition (2.20*).

Theorem 2. If the conditions of the Theorem 1 are met

1) the maximal “strong-type” estimate holds

$$\|\sup_{t>0}|\widetilde{U}_t(f)|\,\|_p\le C_p\,\|f\|_p,\quad p>1;$$

2) in the case $p = 1$ the “strong-type” estimate is replaced by the following relation:

$$\sup_{t>0}\|\widetilde{U}_t(f)\|_1\le C\,\|f\|_1\ ;$$

and the “weak type’ estimate

$$\operatorname{mes}\left\{x\in(0;l)\,\Big|\,\sup_{t>0}|\widetilde{U}_t(f)|\ge\zeta>0\right\}\le\frac{C}{\zeta}\|f\|_1\ ;$$

3) the relation

$$\lim_{t\to+0}\widetilde{U}_t(f,x;v)=f(x) \tag{3.7*}$$

holds for any even function $f\in L^p$ almost everywhere on (0, l), in the metric of the corresponding functional space, at each point of continuity of the function $f(x)$ and, in the case of an everywhere continuous function – uniformly in x.

In particular, the requirement (2.20) is satisfied almost everywhere and at each point of continuity of the function $f(x)$.

Proof of Theorem 2. The basis of the proof is the transformation of $\widetilde{U}_t(f,x;v)$ (see (3.1)) to the corresponding convolution operator with a Fejér-type kernel [16, vol. 1, p. 148]. Let us divide the proof into several steps. Without loss of generality, we can obviously take $l=\pi$.

1. ***Estimates of Partial Sums.*** By virtue of the definition of the coefficients $b_k(f)$, the integral kernel of the partial sums

$$\widetilde{s}_n[f,x]=\sum_{k=1}^{n}b_k(f)\cos\left(k-\frac{1}{2}\right)x,\quad n=1,2,\ldots \tag{3.8}$$

of a series of the type (2.14), in which sines are replaced by cosines, can be brought to the form

$$\sum_{k=1}^{n}\cos\left(k-\frac{1}{2}\right)x\cos\left(k-\frac{1}{2}\right)\tau=$$

$$=\frac{1}{2}\sum_{k=1}^{n}\left(\cos\left(k-\frac{1}{2}\right)(x-\tau)+\cos\left(k-\frac{1}{2}\right)(x+\tau)\right)=$$

$$=\frac{1}{2}D_n(x-\tau)+\frac{1}{2}D_n(x+\tau)\,; \tag{3.9}$$

where

$$D_n(\tau)=\sum_{k=1}^{n}\cos\left(k-\frac{1}{2}\right)\tau$$

is the analogue of the classical Dirichlet kernel [16, vol. 1, p. 86]. Furthermore,

$$D_n(\tau)=\frac{1}{2\sin\frac{\tau}{2}}\sum_{k=1}^{n}(\sin k\tau-\sin(k-1)\tau)=\frac{\sin n\tau}{2\sin\frac{\tau}{2}}. \tag{3.10}$$

The right-hand side of (3.9) (see also (3.10)) has the property of evenness with respect to the variable τ, and therefore its product with $f(\tau)$ will be even. Therefore, the corresponding integral of this product over $(0,\pi)$ can be replaced by half of the integral over $(-\pi,\pi)$, and, due to the evenness of the function f, the partial sums (3.8) take the form

$$\widetilde{s}_n[f,x]=\frac{1}{\pi}\int_{-\pi}^{\pi}f(\tau)\left(\frac{1}{2}D_n(x-\tau)+\frac{1}{2}D_n(x+\tau)\right)d\tau=$$

$$=\frac{1}{\pi}\int_{-\pi}^{\pi}f(\tau)D_n(x-\tau)d\tau=$$

$$=\frac{1}{\pi}\int_{-\pi}^{\pi}f(\tau)\frac{\sin n(x-\tau)}{2\sin\frac{x-\tau}{2}}d\tau,\quad n=1,2,\ldots$$

It is now convenient to replace $f(\tau)$ in the integrand with a function $f_0(\tau)$, which coincides with $f(\tau)$ on $(-\pi,\pi)$ and is then extended in a 2π-periodic manner. This type of periodicity allows without changing the limits of integration to replace the difference $x-\tau$ with a new variable:

$$\widetilde{s}_n[f,x]=\frac{1}{\pi}\int_{-\pi}^{\pi}f(\tau)D_n(x-\tau)d\tau=$$

$$=\frac{1}{\pi}\int_{-\pi}^{\pi}f_0(x-\tau)\frac{\sin n\tau}{2\sin\frac{\tau}{2}}d\tau,\quad n=1,2,\ldots \tag{3.11}$$

For the integral kernel in (3.11), the following estimate is obvious

$$D_n(\tau)=O\left(\min\left\{n;\frac{1}{|\tau|}\right\}\right).$$

Next, we can refer to exactly the same estimate for the classical Dirichlet kernel

$$D_n^{\bullet}(\tau)=\frac{\sin\left(n+\frac{1}{2}\right)\tau}{2\sin\frac{1}{2}\tau},$$

which allows us to bound the corresponding convolution operator from above by the expression $o_x(\ln n)$; see [16, vol. 1, pp. 113–114]. Consequently, the representation (3.11) of the partial sums allows us to assert that $\widetilde{s}_n[f,x]=o_x(\ln n)$ holds almost everywhere, and then, by virtue of (3.5), $\widetilde{s}_n[f,x]v_n(t)\to 0$ when $n\to\infty$ for almost every *x*.

2. ***Estimate of*** $\widetilde{U}_t(f,x,v)$. According to Abel's transformation

$$\widetilde{U}_t(f,x;v)=\lim_{N\to+\infty}\left(\sum_{k=1}^{N-1}\Delta v_k(t)\widetilde{s}_k[f,x]+\widetilde{s}_N[f,x]v_N(t)\right)=$$

$$=\lim_{N\to+\infty}\left(\sum_{k=1}^{N-2}(k+1)\Delta^2 v_k(t)\sigma_k[f,x]+N\Delta v_{N-1}(t)\sigma_{N-1}[f,x]\right). \tag{3.12}$$

Here

$$\sigma_k[f,x]=\frac{1}{k+1}\sum_{\mu=1}^{k}\widetilde{s}_\mu[f,x],\quad k=1,2,...$$

– an analogue of the classical Fejér averages of a Fourier series. In integral form

$$\sigma_k[f,x]=\frac{1}{\pi}\int_{-\pi}^{\pi}f_0(\tau)F_k(x-\tau)d\tau=\frac{1}{\pi}\int_{-\pi}^{\pi}f_0(x-\tau)F_k(\tau)d\tau,\quad k=1,2,... \tag{3.13}$$

In this case

$$F_k(\tau)=\frac{1}{k+1}\sum_{\mu=1}^{k}D_\mu(\tau)=$$

$$=\frac{1}{4(k+1)\sin^2\frac{\tau}{2}}\sum_{\mu=1}^{k}\left(\cos\left(\mu-\frac{1}{2}\right)\tau-\cos\left(\mu+\frac{1}{2}\right)\tau\right)=$$

$$=\frac{\sin\frac{k}{2}\tau\,\sin\frac{k+1}{2}\tau}{2(k+1)\sin^2\frac{\tau}{2}},\quad k=0,1,... \tag{3.14}$$

The kernel (3.14) in (3.13) possesses obvious estimates

$$F_k(\tau)=O\left(\min\left\{k;\frac{1}{(k+1)\tau^2}\right\}\right),$$

in this case, as is well known, the corresponding integral operator (3.13) can be bounded from above (uniformly in *k*) by the Hardy–Littlewood maximal function

$$f_0^*(x)=\sup_{h>0}\frac{1}{h}\int_{x-h}^{x+h}|f_0(\tau)|\,d\tau$$

so that

$$\sup_k|\sigma_k[f,x]|\le C f_0^*(x);$$

see, for example [16, vol. 1, p. 151; 14]. Taking into account the properties (see [14]) of the piecewise convex sequence $\{v_k(t)\}$

$$\sum_{k=1}^{\infty}(k+1)\left|\Delta^2 v_k(t)\right|=O(1)$$

and $N\Delta v_N(t)\to 0$ when $N\to\infty$, from (3.12) we obtain that the estimate via the maximal function also holds for $\widetilde{U}_t(f,x,v)$ uniformly in *t*:

$$\sup_{t>0}|\widetilde{U}_t(f,x;v)|\le C f_0^*(x). \tag{3.15}$$

3. ***Maximal estimates.*** The inequalities contained in statements 1 and 2 of Theorem 2 (the strong-type and weak-type estimates) are, as is well known, also valid for $f_0^*(x)$ (written in place of $\sup_{t>0}|\widetilde{U}_t(f,x;v)|$); see [16, vol. 1, p. 58–59], see also [14]; for example,

$$\|f_0^*\|_p\le C_p\|f_0\|_p=C_p\|f\|_p,\quad p>1.$$

Consequently, the strong and weak estimates for $\sup_{t>0}|\widetilde{U}_t(f,x;v)|$, presented in statements 1 and 2, are direct corollaries of (3.15). As for the inequality for $\sup_{t>0}\|\widetilde{U}_t(f)\|_1$, it follows in the standard way from the uniform boundedness (with respect to *t*) of the family of Lebesgue constants

$$\frac{1}{\pi}\int_0^{\pi}|K_t(\tau)|\,d\tau\le C,$$

where $K_t(\tau)$ is the integral kernel of the operator (3.12); see (3.12)–(3.14). Such boundedness is established in a manner analogous to the estimate (3.15).

The statement (3.7) on convergence at the points of continuity of the function $f(x)$, and in the case of an everywhere continuous function – uniform convergence, follows from the Banach-Steinhaus theorem if we again take into account the boundedness of the corresponding Lebesgue constants as a family of operator norms $U_t(f, x; v)$ acting in the space of continuous periodic functions.

By virtue of the same theorem and the boundedness of the $L^p(p \geq 1)$-norm of the operator $\widetilde{U}_t(f)$ (see §§1–2 of Theorem 2) we obtain the convergence of (3.7) in the corresponding metric; see also [14, 17].

The almost everywhere convergence of (3.7) follows from the weak-type estimate (established in §2 of Theorem 2) and the L^p-convergence in the standard way (see [16, vol. 2, pp. 464–465]). The theorem is proved.

4. Conclusion

A mathematical model of plant material extraction processes has been investigated. This model is a special case of a universal model describing heat and mass transfer processes. The solution to the corresponding boundary value problem is obtained in terms of exponential averages of a Fourier series generated by a continuous periodic function. In turn, the family of exponential averages turns out to be a means of implementing the initial condition. A theorem of corresponding convergence is given. The issues of summability of series over cosine and sine systems arising in the solution of the heat and mass transfer problem are investigated. In particular, the exponential summability of Fourier expansions of functions is established almost everywhere, at points of continuity, as well as in the metric of the corresponding functional space.

It has been established that, under certain conditions, the presented mathematical model converges to the initial physical system that describes the processes of plant material extraction. As a result, this study provides a fundamental scientific description of the mathematical model of plant material extraction processes and examines in detail the properties and approximation capabilities of this model.

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6. Conflict of interests

The authors declare no conflict of interest.

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